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Stable Coalition Structures with Unidimensional Set of Alternatives*

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Abstract

We identify a class of economic and political environments that admit an "$\mathcal{S}$-equilibrium", where both *free mobility* and *free entry* are allowed. We further study the relationship between the set of $\mathcal{S}$-equilibria and the sets of *strong* and *coalition-proof Nash equilibria* in the non-cooperative coalition formation game, which is closely related to the one originally studied by von Neumann and Morgenstern (1944). We also show that the set of $\mathcal{S}$-equilibria induces a proper subset of the coalition structure core of the corresponding non-sidepayment cooperative game in characteristic function form.

Keywords : coalition structure core, strong and coalition-proof Nash equilibrium, $\mathcal{S}$-equilibrium.

JEL classification number: 026.

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1 Introduction

In most social and economic situations the set of alternatives available to a single individual is rather limited. This fact is one of the important reasons for coalition formation. For example, individuals form communities in order to share the costs of the production of local public goods, or workers join a labour union in order to attain a better working contract. This does not, of course, mean that the "optimal size" of a coalition is the entire society, but neither is it the case that individuals operate on their own. In many social and economic activities, individuals form a partition, or a *coalition structure*, where each individual belongs to one and only one coalition.

It is evident that the analysis of coalition formation is important and, in fact, central to the social and behavioral sciences. This analysis, however, proved to be rather difficult, which accounts for the relatively slow progress in this area. In particular, there is only a relatively small number of results that guarantee the existence of a "stable" coalition structure. (For a survey of such results and the reasons for their scarcity, see Greenberg (1991b).) This paper contributes to this line of research. It points out a large class of interesting social environments that admit a "stable" coalition structure.

The model analyzed in this paper belongs to the following rather general framework. Denote the set of individuals by N and the set of all possible alternatives by Ω. Each individual has a utility function over Ω. The feasibility constraints are given by the correspondence ϕ which assigns to each *coalition* (i.e., a nonempty subset of N), C, a subset of Ω, denoted by $\phi(C)$, which consists of alternatives available to members of C, if and when C forms.[1]

Note that this framework includes, in particular, all *simple games*. Indeed such games are characterized by a set of *winning coalitions*, W. That is, the

[1]The feasibility correspondence ϕ might, for example, represent the *effectivity function* (Moulin/ Peleg (1982)), which, though in a different context, determines for each coalition C the subset of Ω that C can enforce as a final outcome regardless the actions of players outside of C.

set of feasible alternatives, $\phi(C)$, available to coalition C is given by:

$$\phi(C) = \begin{cases} \Omega & if \;\; C \;\; belongs \;\; to \;\; W \\ \emptyset & if \;\; C \;\; does \;\; not \;\; belong \;\; to \;\; W. \end{cases}$$

Thus, a winning coalition can choose, for its members, any alternative in Ω it desires, while a non-winning coalition "has no say" whatsoever.

Within this framework we define our notion of equilibrium, which shares the following two properties:

Free mobility: Each individual is free to join the coalition which adopts the alternative he likes best from those offered by the *existing* coalitions.

Free entry: Every coalition C is free to form (and then choose any alternative from the set $\phi(C)$ its members desire).

More specifically, in equilibrium, individuals form a partition P of N, where each coalition $C \in P$ chooses an alternative $a(C)$ in $\phi(C)$ in such a way that each individual belongs to the coalition whose alternative he likes best among the offered alternatives. Moreover, there is no coalition T and an alternative $\omega \in \phi(T)$ which makes all members of T better off than they currently are. Thus, our equilibrium notion borrows from two important game-theoretic solution concepts: The Nash equilibrium (reflected by *free mobility*) and the core (reflected by *free entry*).

It is obvious that some restrictions on Ω and ϕ are necessary if we are to obtain the existence of an equilibrium. We impose the following restrictive, though frequently employed, assumptions: The set of alternatives Ω is unidimensional; the utility functions of the individuals are single-peaked; and the correspondence ϕ is (weakly) monotone, that is, if an alternative ω is feasible for a coalition C, then it also feasible for all coalitions that contain C.

We prove that under these assumptions, whenever the set of individuals and/or the set of alternatives contains a finite number of elements, an equilibrium exists. We also show that if both N and Ω contain an infinite number of elements then an equilibrium may fail to exist. But, following one of the anonymous referee's suggestion, we prove that if the utility functions of the individuals are equicontinuous, then every society that can be represented as a simple game admits an "ε-equilibrium" for all positive ε (also when both sets Ω and N are infinite).

Our results considerably generalize Greenberg/ Weber (1985), who established the existence of "a stable multiparty equilibrium under the fixed standard method": In equilibrium, every incumbent is supported by at least m voters, while a potential entrant can attract no more than $m-1$ voters, where m is some *a priori* given quota. To see that this model is a special case of the one we study in this paper, define the set of winning coalitions, W, to consist of all coalitions (constituencies) with at least m voters.

As mentioned above, our equilibrium notion is more demanding than both the core and the Nash equilibrium. We demonstrate this assertion by considering two associated games. The first (see Section 4) is a cooperative game in characteristic function form whose coalition structure core (Aumann/ Dreze (1974)) strictly includes the set of our equilibria. Our existence result yields, therefore, that the coalition structure core of this game is nonempty.

The second game we associate with our model is a noncooperative coalition formation game in normal (or strategic) form, which is related to the one originally introduced by von Neumann and Morgenstern (1944).[2] It turns out that our equilibrium coincides with the strong Nash equilibrium, which, in turn, also coincides with the coalition proof Nash equilibrium in this game. Using Greenberg (1989) and (1991a) this result implies that the set of S-equilibria is both "internally and externally stable". In particular, any non S-equilibrium can be "defeated" by an S-equilibrium is a "sub-society".

The paper is organized as follows: In the next section we present the model, the formal definitions, and state our main result (Theorem 1) on the existence of an equilibrium for societies with either a finite set of individuals

[2]Other modifications of the coalition formation game were considered by Kalai/ Postlewaite/ Roberts (1979) and Hart/ Kurz (1983).

or a finite number of alternatives. In Section 3 we present the proof of Theorem 1 (which, although elementary, is quite involved). Section 4 is devoted to studying the relationship between the set of our equilibria and the coalition structure core for the associated game in characteristic function form. In Section 5 we show the equivalence of the set of our equilibria and the sets of strong and of coalition-proof Nash equilibria in the normal form coalition formation game. In Section 6 we demonstrate that the existence of an equilibrium (Theorem 1) cannot be extended for the case when both the set of players and the set of alternatives are infinite. We provide, however, sufficient conditions which guarantee the existence, for all positive ε, of an ε-equilibrium for all societies that can be represented as simple games.

2 The Model

Let Ω denote the universe of potential alternatives. There are n individuals given by the set N. Each individual $i \in N$ has a preference ordering over elements of Ω which is represented by the utility function $u_i : \Omega \rightarrow R$. For each coalition C, a subset $\phi(C)$ of Ω denotes the set of feasible alternatives from which C can choose if and when it forms. (Note that we allow for $\phi(C)$ to be empty, in which case members of C have no power concerning what alternative will be adopted. Another interpretation of $\phi(C) = \emptyset$ is that C is (legally) not allowed to form.) A full description of society, is, therefore, provided by the quadruple $\mathcal{S} = (N, \Omega, \phi, U)$, where $U = \{u_i\}_{i \in N}$ is the profile of the players' preferences.

Within this framework we are interested to know what coalitions will form and which alternatives will they adopt. The answer to this central question depends, of course, on the restrictions imposed on players' mobility. Indeed, if no individual can "vote with his feet", i.e., if every player must remain in the coalition to which he is assigned, then every coalition structure[3] is

[3]A collection P of pairwise disjoint coalitions $C_1, C_2, ..., C_J$, is called a *partition* or a

"stable". We, in contrast, explore the existence of a stable partition with both free mobility and free entry.

Definition 1: Let $\mathcal{S} = (N, \Omega, \phi, U)$ be a society. A pair (P, A) is called an $\mathcal{S}-$ *equilibrium* if

(D.1.1) $P = \{C_1, C_2, ..., C_J\}$ is a (finite) partition of N.

(D.1.2) $A = \{a_1, a_2, ..., a_J\}$ with $a_j \in \phi(C_j)$ for all $j = 1, ..., J$.

(D.1.3) For all $j, h = 1, 2, .., J$, $u_i(a_j) \geq u_i(a_h)$ whenever $i \in C_j$.

(D.1.4) There exist no $C \subset N$ and $\omega \in \phi(C)$ such that $u_i(\omega) > u_i(a)$ for all $i \in C$ and all $a \in A$.

(D.1.1) and (D.1.2) define the partition of individuals and the alternatives chosen by each coalition in that partition, a choice that must, of course, be feasible. (D.1.3) is a Nash-type condition: No individual can be made better off by migrating to another (existing) coalition. Finally, (D.1.4) is a core-type condition: There is no group of individuals who can form a coalition and choose an alternative which makes each of its members better off than they currently are.

In view of the demanding nature of our equilibrium, it is evident that some restrictions must be imposed on a society $\mathcal{S}$ in order for it to admit an equilibrium. Perhaps somewhat surprisingly, the following three relatively mild and frequently employed assumptions, suffice to this end:

Assumption (A.1) - Unidimensionality: Ω is a compact subset of the real line $\Re$.

Assumption (A.2) - Monotonicity: If $C \subset T \subset N$ and $\omega \in \phi(C)$, then $\omega \in \phi(T)$. Moreover, $\phi(N)$ is nonempty and for each $C \subset N$, $\phi(C)$ is a closed subset of Ω.

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coalition structure of N if $\bigcup_{j=1}^{J} C_j = N$. The set of all partitions of N is denoted by $\mathcal{P}$.

Assumption (A.3) - Single-peakedness: For each individual $i \in N$, u_i is continuous and single-peaked. That is, for each $i \in N$ there exists an alternative $p^i \in \Omega$, called i's *peak*, such that $u_i(p^i) \geq u_i(\omega_1) \geq u_i(\omega_2)$ when either $p^i \geq \omega_1 \geq \omega_2$ or $p^i \leq \omega_1 \leq \omega_2$.[4]

Assumption (A.1) restricts the set of potential alternatives to a union of closed intervals of the real line. Assumption (A.2) states that if an alternative a is feasible for a coalition C, then it is also feasible for every group of individuals that contains C. Assumption (A.3) restricts, in a classical way, the domain of individuals' preferences.

Our main result concerning the existence of an $\mathcal{S}$-equilibrium is:

Theorem 1: Every society $\mathcal{S}$ that satisfies (A.1) - (A.3), and where either N and/or Ω is a finite[5] set, admits an $\mathcal{S}$-equilibrium.

The next section is devoted to the proof of Theorem 1, which, although elementary, (the basic technique being induction), is rather long. The reader who is not interested in mathematical details can proceed directly to Section 4.

3 Proof of Theorem 1

Assumption (A.2) immediately yields the following two useful observations. First, we may assume, without loss of generality, that $\phi(N) = \Omega$. Indeed, every alternative in Ω which does not belong to $\phi(N)$ is not feasible for any coalition, and hence, cannot be part of an equilibrium. Second, for the purpose of establishing the existence of an $\mathcal{S}$-equilibrium, we may assume that if (P, A) is an $\mathcal{S}$-equilibrium, then A consists of *distinct* alternatives. Indeed, let (P, A) be an $\mathcal{S}$-equilibrium. Denote by $\{A\}$ the set of

[4] If Ω is a convex set, this assumption is equivalent to quasi-concavity of the utility functions.

[5] See Section 5 for a discussion on possible extensions of Theorem 1 to the case where both N and Ω are infinite.

distinct alternatives in A and for each $a \in \{A\}$ denote by C^a the union of all coalitions that adopt alternative a, i.e., $C^a = \bigcup_{\{j|a_j=a\}} C_j$. Then, clearly, the pair $(\hat{P}, \{A\})$, where $\hat{P} = \{C^a\}_{a \in \{A\}}$, constitutes an $\mathcal{S}$-equilibrium.

Let $E(\mathcal{S})$ denote the set of all $\mathcal{S}$-equilibria such that $(P, A) \in E(\mathcal{S})$ implies that A consists of distinct alternatives, i.e., $A = \{A\}$. Let $\lambda(\mathcal{S})$ denote the set of all subsets A of Ω for which there exists a partition P so that $(P, A) \in E(\mathcal{S})$. That is,

$$\lambda(\mathcal{S}) = \{A \subset \Omega \mid \exists\, P \in \mathcal{P} \ \textit{such that} \ (P, A) \in E(\mathcal{S})\}.$$

In order to prove Theorem 1, we shall first consider societies where the preferences of each individual are strict. That is, we impose the following additional condition:

Assumption (A.4) - Strict preferences: For each individual $i \in N$, and for any two distinct alternatives $\omega_1, \omega_2 \in \Omega$, $u_i(\omega_1) \neq u_i(\omega_2)$.

Assumption (A.4) implies that for each individual $i \in N$ i's peak p^i is unique and, moreover, $u_i(p^i) > u_i(\omega_1) > u_i(\omega_2)$ when either $p^i > \omega_1 > \omega_2$ or $p^i < \omega_1 < \omega_2$. Another implication of this additional assumption is that for societies that satisfy (A.4), if $A \in \lambda(S)$ then every $a \in A$ is feasible (according to ϕ) for those individuals who *strictly* prefer a over any other alternative in A. That is, $a \in \phi(S(a; A, \mathcal{S}))$, where

Definition 2: Let $\mathcal{S} = (N, \Omega, \phi, U)$ be a society and let Q be a subset of Ω. The *support* of alternative $\omega \in \Omega$, denoted $S(\omega; Q, \mathcal{S})$, is the set of individuals who strictly prefer ω over any other alternative in Q. That is,

$$S(\omega; Q, \mathcal{S}) = \{i \in N \mid u_i(\omega) > u_i(q) \ \ \forall q \in Q, q \neq \omega\}.$$

The proof of Theorem 1 is based on the following

Proposition: Let $\mathcal{S} = (N, \Omega, \phi, U)$ satisfy (A.1) - (A.4). If both N and Ω are finite sets, then the set $E(\mathcal{S})$ is nonempty.

The proof of this proposition benefits from the following five observations. Let $\mathcal{S}$ be a society that satisfies (A.1) - (A.4) and where both N and Ω are finite sets. Let Q denote a nonempty set of alternatives, i.e., $Q = \{q_1, ..., q_T\}$ with $q_1 < \cdots < q_T$. Then:

O.1: Let $\omega \in \Omega$, $q_t \in Q$ and $i \in N$. If either $p^i \geq q_t > \omega$ or $p^i \leq q_t < \omega$ then $i \notin S(\omega; Q, \mathcal{S})$:
Indeed, in both cases assumptions (A.3)-(A.4) imply that $u_i(q_t) > u_i(\omega)$. Hence, $i \notin S(\omega; Q, \mathcal{S})$.

O.2: Let $\omega \in \Omega$, and $q_t, q_{t+1} \in Q$ be such that $q_t < \omega < q_{t+1}$. Then $S(\omega; Q, \mathcal{S}) = S(\omega; \{q_t, q_{t+1}\}, \mathcal{S})$:
Indeed, denote $\hat{N} = \{i \in N \mid q_t < p^i < q_{t+1}\}$. Then, for all $i \in \hat{N}$, $u_i(q_t) > u_i(q_h)$ if $h < t$ and $u_i(q_{t+1}) > u_i(q_r)$ if $r > t+1$. By O.1, $q_t < \omega < q_{t+1}$ implies that $S(\omega; Q, \mathcal{S}) = S(\omega; Q, \mathcal{S}) \bigcap \hat{N} = S(\omega; \{q_t, q_{t+1}\}, \mathcal{S})$.

O.3: Let $\alpha_1 \leq \alpha_2 < \alpha_3 \leq \alpha_4$, $\alpha_i \in \Omega$, $i = 1, 2, 3, 4$. Then for all ω, where $\alpha_2 < \omega < \alpha_3$, $S(\omega; \{\alpha_2, \alpha_3\}, \mathcal{S}) \subset S(\omega; \{\alpha_1, \alpha_4\}, \mathcal{S})$:
Indeed, let $i \in S(\omega; \{\alpha_2, \alpha_3\}, \mathcal{S})$. Then, by (A.3), $\alpha_2 \leq p^i \leq \alpha_3$. Thus, by (A.3)-(A.4), $u_i(\omega) > \max\{u_i(\alpha_1), u_i(\alpha_4)\}$, yielding $i \in S(\omega; \{\alpha_1, \alpha_4\}, \mathcal{S})$.

O.4: Let $\alpha_1 < \alpha_2 \leq \alpha_3$, $\alpha_i \in \Omega$, $i = 1, 2, 3$. Then $S(\alpha_3; \{\alpha_1, \alpha_3\}, \mathcal{S}) \subset S(\alpha_2; \{\alpha_1, \alpha_2\}, \mathcal{S})$ and $S(\alpha_1; \{\alpha_1, \alpha_2\}, \mathcal{S}) \subset S(\alpha_1; \{\alpha_1, \alpha_3\}, \mathcal{S})$:
Indeed, let $i \in S(\alpha_3; \{\alpha_1, \alpha_3\}, \mathcal{S})$. Then $u_i(\alpha_3) > u_i(\alpha_1)$. If $p^i < \alpha_2$ then, by (A.3)-(A.4), $u_i(\alpha_2) > u_i(\alpha_3)$, which implies $u_i(\alpha_2) > u_i(\alpha_1)$ or $i \in S(\alpha_2; \{\alpha_1, \alpha_2\}, \mathcal{S})$. If $p^i \geq \alpha_2$ then, by (A.3)-(A.4), $u_i(\alpha_2) > u_i(\alpha_1)$, yielding $i \in S(\alpha_2; \{\alpha_1, \alpha_2\}, \mathcal{S})$.

Similarly, consider $i \in S(\alpha_1; \{\alpha_1, \alpha_2\}, \mathcal{S})$. Then, $u_i(\alpha_1) > u_i(\alpha_2)$ and $p^i < \alpha_2$. Thus, $u_i(\alpha_1) > u_i(\alpha_2) > u_i(\alpha_3)$, implying $i \in S(\alpha_1; \{\alpha_1, \alpha_3\}, \mathcal{S})$.

In order to state the last observation we need some additional notation. Consider a society $\mathcal{S} = (N, \Omega, \phi, U)$ and let $\tilde{\Omega} \subset \Omega$. We denote the restriction of $\mathcal{S}$ to $\tilde{\Omega}$ by:

$$\mathcal{S}/\tilde{\Omega} = (N, \tilde{\Omega}, \tilde{\phi}, \tilde{U})$$

where $\tilde{\phi}(C) = \phi(C) \bigcap \tilde{\Omega}$ for all $C \subset N$, and $\tilde{u}_i(\omega) = u_i(\omega)$ for all $\omega \in \tilde{\Omega}$ and all $i \in N$.

O.5: Let $\omega \in \Omega$. Denote: $\tilde{\Omega} = \Omega \setminus \{\omega\}$. Let $Q \subset \tilde{\Omega}$ be such that $Q \in \lambda(\mathcal{S}/\tilde{\Omega})$. Then $Q \notin \lambda(\mathcal{S})$ if and only if $\omega \in \phi(S(\omega; Q, \mathcal{S}))$:
This observation follows immediately from the fact that $\phi(S(q; Q, \mathcal{S})) = \tilde{\phi}(S(q; Q, \mathcal{S}/\tilde{\Omega}))$ for all $q \in \Omega \setminus \{\omega\}$.

Proof of the Proposition: The proof is by induction on the number of alternatives, H, in Ω. For $H = 1$ we have $p^i = \omega_1$ for all $i \in N$. Since, by (A.2), $\omega_1 \in \phi(N)$, it follows that $\{\omega_1\} \in \lambda(\mathcal{S})$.

The induction hypothesis is that the Proposition holds for any set of alternatives with no more than $H - 1$ alternatives, $H \geq 2$. We have to show that it also holds for any set with H alternatives. Let $\Omega = \{\omega_1, \omega_2, ..., \omega_H\}$ be such a set, where $\omega_1 < \omega_2 < \cdots < \omega_H$. Denote

$$F = \{i \in N \mid p^i = \omega_H\}.$$

$$\Omega^H = \Omega \setminus \{\omega_H\}, \;\; \mathcal{S}^H = \mathcal{S}/\Omega^H;$$

$$\Omega^{H-1} = \Omega \setminus \{\omega_{H-1}\}, \;\; \mathcal{S}^{H-1} = \mathcal{S}/\Omega^{H-1}.$$

Observe that since $\mathcal{S}$ satisfies assumptions (A.1)-(A.4), so do societies $\mathcal{S}^H$ and $\mathcal{S}^{H-1}$. Since the cardinality of Ω^H and Ω^{H-1} is equal to $H - 1$, by the induction hypothesis both $\lambda(\mathcal{S}^H)$ and $\lambda(\mathcal{S}^{H-1})$ are nonempty. Let

$$A = \{a_1, ..., a_J\} \in \lambda(\mathcal{S}^H),$$

where $a_1 < \cdots < a_J$. If $\omega_H \notin \phi(S(\omega_H; A, \mathcal{S}))$ then, by O.5, $A \in \lambda(\mathcal{S})$, and we are done. Assume, therefore, that

$$\omega_H \in \phi(S(\omega_H; A, \mathcal{S})). \tag{1}$$

Lemma: If either $\{\omega_H\} \in \lambda(\mathcal{S}^{H-1})$ or $\{\omega_{H-1}\} \in \lambda(\mathcal{S}^H)$ then the set $\lambda(\mathcal{S})$ is nonempty.

Proof of the Lemma: Distinguish between the following two cases:

(α) $\omega_{H-1} \in \phi(N \setminus F)$: Since either $\{\omega_H\} \in \lambda(\mathcal{S}^{H-1})$ or $\{\omega_{H-1}\} \in \lambda(\mathcal{S}^H)$ then $\omega \notin \phi(S(\omega; \{\omega_{H-1}, \omega_H\}, \mathcal{S}))$ for all $\omega \in \Omega$ with $\omega < \omega_{H-1}$. Moreover, $S(\omega_{H-1}; \{\omega_{H-1}, \omega_H\}, \mathcal{S}) = N \setminus F$, $S(\omega_{H-1}; \{\omega_{H-1}\}, \mathcal{S}) = N \setminus F$, and $S(\omega_H; \{\omega_{H-1}, \omega_H\}, \mathcal{S}) = F$. It follows that if $\omega_H \in \phi(F)$ then $\{\omega_{H-1}, \omega_H\} \in \lambda(\mathcal{S})$, and if $\omega_H \notin \phi(F)$, then $\{\omega_{H-1}\} \in \lambda(\mathcal{S})$.

(β) $\omega_{H-1} \notin \phi(N \setminus F)$: Since $S(\omega_{H-1}; \{\omega_H\}, \mathcal{S}) = N \setminus F$ it follows that $\omega_{H-1} \notin \phi(S(\omega_{H-1}; \{\omega_H\}, \mathcal{S})$. Hence, by O.5, if $\{\omega_H\} \in \lambda(\mathcal{S}^{H-1})$ then $\{\omega_H\} \in \lambda(\mathcal{S})$, and we are done. Consider, therefore, the case where $\{\omega_H\} \notin \lambda(\mathcal{S}^{H-1})$. Then, by the conditions of the Lemma, $\{\omega_{H-1}\} \in \lambda(\mathcal{S}^H)$. If $\omega_H \notin \phi(F)$ then, by O.5, $\{\omega_{H-1}\} \in \lambda(\mathcal{S})$. Assume, therefore, that $\omega_H \in \phi(F)$. Since $\{\omega_H\} \notin \lambda(\mathcal{S}^{H-1})$, there exists an alternative $a \leq \omega_{H-1}$ such that $a \in \phi(S(a; \{\omega_H\}, \mathcal{S}))$. Let a be such that $a \geq b$ for all $b < \omega_H$ satisfying $b \in \phi(S(b; \{\omega_H\}, \mathcal{S}))$. We shall show that $\{a, \omega_H\} \in \lambda(\mathcal{S})$. Since $a \in \phi(S(a; \{a, \omega_H\}, \mathcal{S}))$ and $\omega_H \in \phi(F)$, it remains to prove that $\omega \notin \phi(S(\omega; \{a, \omega_H\}, \mathcal{S}))$ for all $\omega \notin \{a, \omega_H\}$.

If $\omega < a$, then by O.4, $S(\omega; \{a, \omega_H\}, \mathcal{S}) \subset S(\omega; \{\omega_{H-1}\}, \mathcal{S}))$. Since $\{\omega_{H-1}\} \in \lambda(\mathcal{S}^H)$, it follows that $\omega \notin \phi(S(\omega; \{a, \omega_H\}, \mathcal{S}))$. If, on the other hand, $a < \omega < \omega_H$, then $S(\omega; \{a, \omega_H\}, \mathcal{S}) \subset S(\omega; \{\omega_H\}, \mathcal{S})$ together with the choice of a imply that $\omega \notin \phi(S(\omega; \{a, \omega_H\}, \mathcal{S}))$. □

To conclude the proof of the Proposition, we shall show that the set $\lambda(\mathcal{S})$ is nonempty in the following two cases:

Case (I): $a_J = \omega_{H-1}$;
Case (II): $a_J < \omega_{H-1}$.

Case (I): Since $a_J = \omega_{H-1}$, $F = S(\omega_H; A, \mathcal{S})$. By (1), therefore,

$$\omega_H \in \phi(F). \tag{2}$$

If $J = 1$, then the validity of the Proposition in this case follows from the

Lemma. Assume, therefore, that $J > 1$. By (2), $\omega_H \in Q$ for all $Q \in \lambda(\mathcal{S}^{H-1})$ (which, by the induction hypothesis, is nonempty). If, moreover, $\{\omega_H\} \in \lambda(\mathcal{S}^{H-1})$, then the validity of the Proposition in this case also follows from the Lemma. Assume, therefore, that $|Q| > 1$ for each $Q \in \lambda(\mathcal{S}^{H-1})$. Let

$$B = \{b_1, b_2, \ldots, b_K = \omega_H\} \in \lambda(\mathcal{S}^{H-1})$$

be such that $b_{K-1} \geq q_{m-1}$ for all $Q \in \lambda(\mathcal{S}^{H-1})$, $Q = \{q_1, q_2, \ldots, q_m\}$. As we shall now verify, this choice of B implies that the second largest alternative in B, b_{K-1}, is no less than the second largest alternative in A, a_{J-1}. That is, we shall show that

$$b_{K-1} \geq a_{J-1}. \tag{3}$$

Indeed, otherwise, the choice of B implies that the set $D = \{a_1, \ldots, a_{J-1}, \omega_H\}$ does not belong to $\lambda(\mathcal{S}^{H-1})$. The fact that $A \in \lambda(\mathcal{S}^H)$ implies, therefore, that there exists $\omega, a_{J-1} < \omega < \omega_H$, such that $\omega \in \phi(S(\omega; D, \mathcal{S}))$. Since we assumed that $a_{J-1} > b_{K-1}$, O.3 implies that $S(\omega; D, \mathcal{S}) \subset S(\omega; B, \mathcal{S})$. By (A.2), $\omega \in \phi(S(\omega; B, \mathcal{S}))$, which, together with that fact that $\omega \notin B$ (recall that $b_{K-1} < a_{J-1} < \omega < \omega_H$), contradict $B \in \lambda(\mathcal{S}^{H-1})$. Thus (3) holds.

Denote by $A^* \equiv A \bigcup \{\omega_H\}$, i.e., $A^* = \{a_1, \ldots, a_{J-1},\ a_J = \omega_{H-1}, \omega_H\}$. We shall conclude the proof of this case by showing that either $B \in \lambda(\mathcal{S})$ or else $A^* \in \lambda(\mathcal{S})$. By O.2, $S(\omega_{H-1}; B, \mathcal{S}) = S(\omega_{H-1}; \{b_{K-1}, \omega_H\}, \mathcal{S})$, and $S(\omega_{H-1}; A^*, \mathcal{S}) = S(\omega_{H-1}; \{a_{J-1}, \omega_H\}, \mathcal{S})$. By (3), therefore,

$$S(\omega_{H-1}; B, \mathcal{S}) \subset S(\omega_{H-1}; A^*, \mathcal{S}). \tag{4}$$

By O.5 we have that if $B \notin \lambda(\mathcal{S})$ then $\omega_{H-1} \in \phi(S(\omega_{H-1}; B, \mathcal{S}))$. Hence, by (4), $\omega_{H-1} \in \phi(S(\omega_{H-1}; A^*, \mathcal{S}))$. Since $A \in \lambda(\mathcal{S}^H)$, O.2, O.3 and (2) yield that $A^* \in \lambda(\mathcal{S})$. Thus, $\lambda(\mathcal{S}) \bigcap \{B, A^*\} \neq \phi$.

Case (II): Consider the society $\hat{\mathcal{S}} = (N, \Omega, \hat{\phi}, U)$ where for each $C \subset N$

$$\hat{\phi}(C) = \begin{cases} \phi(C) \bigcup \{\omega_{H-1}\} & if\ \ F \subset C \\ \phi(C) & otherwise. \end{cases}$$

Clearly, $\hat{\mathcal{S}}$ satisfies assumption (A.2). Moreover, since $\omega_{H-1} \in \hat{\phi}(F)$, it follows that ω_{H-1} belongs to each $\hat{A} \in \lambda(\hat{\mathcal{S}}^H)$. Thus, as shown in Case (I), the set $\lambda(\hat{\mathcal{S}})$ is nonempty. Let $\hat{Q} \in \lambda(\hat{\mathcal{S}})$. We shall end the proof by showing that $\hat{Q} \in \lambda(\mathcal{S})$. Indeed, (1) and the definition of $\hat{\mathcal{S}}$ imply that $\omega_H \in \hat{\phi}(F)$. Thus, $\omega_H \in \hat{Q}$. Hence, for each $\omega \neq \omega_H$, $S(\omega, \hat{Q}, \mathcal{S}) \bigcap F = \emptyset$ and therefore, $\phi(S(\omega, \hat{Q}; \mathcal{S})) = \hat{\phi}(S(\omega, \hat{Q}; \hat{\mathcal{S}}))$. That is, $\hat{Q} \in \lambda(\hat{\mathcal{S}})$ implies $\hat{Q} \in \lambda(\mathcal{S})$. □

The Proposition that we just proved yields the following two claims which conclude the proof of Theorem 1.

Claim 1: Let $\mathcal{S} = (N, \Omega, \phi, U)$ satisfy (A.1) - (A.3), where Ω is a finite set. Then the set $E(\mathcal{S})$ is nonempty.

Claim 2: Let $\mathcal{S} = (N, \Omega, \phi, U)$ satisfy (A.1) - (A.3), where N is a finite set. Then the set $E(\mathcal{S})$ is nonempty.

Proof of Claim 1: Let Ω be a finite set. Then there is a finite number, k, of complete, transitive and strict orderings over Ω. Let $\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_k\}$ denote the set of all such orderings. For $\sigma_t \in \sum$, and $\omega, \tilde{\omega} \in \Omega$, we write $\omega\sigma_t\tilde{\omega}$ if, according to σ_t, ω is (strictly) preferred to $\tilde{\omega}$. Denote by F_t the set of individuals whose preferences agree with σ_t, with ties being broken by the ranking of the alternatives in Ω. (Recall that Ω is a subset of $\Re$, hence for any two alternatives $\omega, \tilde{\omega} \in \Omega$, either $\omega < \tilde{\omega}$ or else $\tilde{w} < \omega$.) That is, $i \in F_t$ if and only if for any two alternatives $\omega, \tilde{\omega} \in \Omega$, the following holds: $\omega\sigma_t\tilde{\omega}$ if and only if either $u_i(\omega) > u_i(\tilde{\omega})$, or else $u_i(\omega) = u_i(\tilde{\omega})$ and $w < \tilde{w}$.

Consider the society $\mathcal{S}^* = (N^*, \Omega, \phi^*, U^*)$ where

$$N^* = \{1, 2, ..., k\};$$

$$U^* = \{u_t\}, \quad t = 1, 2, ..., k;$$

$$\phi^*(T) = \phi(\bigcup_{t \in T} F_t) \ \forall \ T \subset N^*$$

Since $\mathcal{S}$ satisfies (A.1) - (A.3), $\mathcal{S}^*$ satisfies (A.1) - (A.4). As Ω and N^* are finite sets, the Proposition yields the existence of $A^* \in \lambda(\mathcal{S}^*)$, and it is easy to verify that $A^* \in \lambda(\mathcal{S})$. $\square$

Proof of Claim 2: Since Ω is a (compact) subset of $\Re$, there exists a countable sequence of alternatives in Ω, $\{\omega_m\}_{m=1}^{\infty}$ which is everywhere dense in Ω. For each $m = 1, 2, ...$ consider the society $\mathcal{S}/\Omega_m$, where $\Omega_m = \{\omega_1, ..., \omega_m\}$. Since $\mathcal{S}/\Omega_m$ satisfies assumptions (A.1) - (A.3) and Ω_m is a finite set for all m, Claim 1 yields the existence of a sequence $(P_m, A_m) \in E(\mathcal{S}/\Omega_m)$, where P_m is a partition of N and A_m is a finite subset of Ω_m. We shall show that there is a subsequence of the sequence $\{(P_m, A_m)\}_{m=1}^{\infty}$ whose limit exists and belongs to $E(\mathcal{S})$.

Since the number of possible partitions of the finite set N is finite, there exists a subsequence of $\{P_m\}$ all of whose members coincide with some fixed partition $\bar{P} = \{C_1, ..., C_J\}$. Also, since Ω is a compact subset of $\Re$, there is a subsequence of the above subsequence, (w.l.o.g., the original sequence itself), such that $A_m = \{a_1^m, ..., a_J^m\} \subset \Omega_m$ and $\lim_{m\to\infty} a_j^m = \bar{a}_j \in \Omega$ for all $j = 1, 2, .., J$. Define $\bar{A} = \{\bar{a}_1, ..., \bar{a}_J\}$. We shall now show that $(\bar{A}, \bar{P})$ is an $\mathcal{S}$-equilibrium. Clearly (D.1.1) is satisfied, and the closedness of $\phi(C_j)$ for each $j = 1, 2, .., J$, yields (D.1.2). Moreover, since $(\bar{P}, A_m) \in E(\mathcal{S}/\Omega_m)$ for all m, and N is a finite set, we have that $(\bar{P}, \bar{A})$ satisfies (D.1.3) and (D.1.4). (Recall that $\bigcup_m \Omega_m$ is everywhere dense in Ω).$\square$

4 Coalition Structure Core

As stated in the Introduction, our equilibrium notion borrows from both the core and the Nash equilibrium. This section formalizes the first assertion. We prove that the set of $\mathcal{S}$-equilibria always yields a subset and, in general, a proper subset of the core of a cooperative game without side payments which we shall define below. One implication of Theorem 1 is, therefore, that the core of this game is nonempty. It is interesting to note that the usual technique for such a proof, namely, showing that the game is balanced (Scarf (1967)), is not applicable in this case. Indeed, as Le Breton (1989) shows, the associated game is not, in general, balanced. (This fact may account for the relatively involved proof of Theorem 1.)

Let $\mathcal{S} = (N, \Omega, \phi, U)$ be a society with a finite number of individuals, given by the set $N = \{1, 2, \ldots, n\}$. In view of the compactness of Ω and the continuity of each u_i, $i \in N$, we may assume, w.l.o.g., that $u_i(\omega) > 0$ for all $i \in N$ and all $\omega \in \Omega$. Since a coalition C is free to choose any alternative from the set $\phi(C)$, it seems most natural to associate with $\mathcal{S}$ the following coalitional form game without side payments (N, V), where the characteristic function of V is given by: [6]

$$V(C) = \begin{cases} \{\mathrm{x} \in \Re^N \mid x^i \leq 0 \;\; \forall i \in C\} & if \;\; \phi(C) = \emptyset, \\ \{\mathrm{x} \in \Re^N \mid \exists\, \omega \in \phi(C) \;\; s.t. \;\; u_i(\omega) \geq x^i \;\; \forall i \in C\} & if \;\; \phi(C) \neq \emptyset. \end{cases}$$

Intuitively, the projection $V(C)$ on C is the set of all C-attainable utility levels. The coalition structure core of the game (N, V) is defined as follows:

Definition 3: Let $P \in \mathcal{P}$ be a partition of N. *The P-core of the game* (N, V) is given by:

$$Core_P V = \{\mathrm{x} \in \bigcap_{C \in P} V(C) \mid \nexists \;\; T \subset N \;\; and \;\; \mathrm{y} \in V(T) \;\; s.t. \;\; \mathrm{y}^T > \mathrm{x}^T\}$$

We say that the game (N, V) has *a nonempty coalition structure core* if there exists a partition $P \in \mathcal{P}$ for which the P-core is nonempty.

Theorem 1 immediately yields:

Theorem 2: Let $\mathcal{S} = (N, \Omega, \phi, U)$ satisfy (A.1) - (A.3). Then the game (N, V) has a nonempty coalition structure core. Moreover, the payoff

[6] Recall that $\Re^N$ denotes the Euclidean space of dimension n, and for $\mathrm{x} \in \Re^N$ and $C \subset N$, x^C denotes the projection of x on C. (In particular, x^i is the i-th coordinate of x.) For $\mathrm{x}, \mathrm{y} \in \Re^C$ we denote $\mathrm{x} > \mathrm{y}$ if $x^i > y^i$ for all $i \in C$.

which results from an $\mathcal{S}$-equilibrium $(P, A) \in E(\mathcal{S})$ belongs to the P-core.

Proof of Theorem 2: Since N is finite, Theorem 1 guarantees that $E(\mathcal{S}) \neq \emptyset$. Let $(P, A) \in E(\mathcal{S})$ and consider the associated payoff x, where for all $i \in N$ and all j with $i \in C_j \in P$, $x^i = u_i(a_j)$. By the definition of the game (N, V), $\mathrm{x} \in \bigcap_{C \in P} V(C)$, and by (D.1.4), there are no $C \subset N$ and $\mathrm{y} \in V(C)$ such that $\mathrm{y}^C > \mathrm{x}^C$. That is, $\mathrm{x} \in Core_P V$. □

We now show that the converse of Theorem 2 need not hold. That is, the coalition structure core of the game (N, V) might strictly include the set of equilibrium payoffs. The reason for such a phenomenon is that it is possible to have a payoff vector which cannot be blocked, but nevertheless there exist players who can benefit by joining another existing coalition, C. The problem is that members of C do not benefit from such a move.

Example 1: Consider the society $\mathcal{S} = (N, \Omega, \phi, U)$, where

$$N = \{1, 2, ..., n\}, \quad \Omega = \{a, b\},$$

$$\phi(C) = \begin{cases} \Omega & if\ either\ 1 \in C\ or\ \{2, 3, ..., n\} \subset C \\ \emptyset & otherwise \end{cases}$$

and $U = \{u_i\}_{i \in N}$ is given by

$$u_i(a) = 2, \quad u_i(b) = 1 \quad for \quad i = 1, 2, ..., n-1,$$

$$u_n(a) = 1, \quad u_n(b) = 2.$$

Thus, a coalition is winning if and only if it contains player 1, or else it includes all the players other than 1. Players $1, 2, ..., n-1$ prefer alternative a over b, whereas the n-th player has the opposite ranking. Clearly, since $\phi(\{1\}) = \Omega$, in equilibrium, player 1 belongs to a coalition that chooses alternative a. This implies that the only equilibrium is (P, A) where $P = \{N\}$

and $A = \{a\}$. That is, the grand coalition adopts alternative a and the resulting payoff is x, where $\mathbf{x} = (2, 2, \ldots, 2, 1)$. Consider, however, the pair $(\hat{P}, \hat{A})$, where $\hat{P} = \{\{1\}, \{2, 3, ..., n\}\}$ and $\hat{A} = \{a, b\}$, which yields the payoff vector $\mathbf{y} = (2, \underbrace{1, ..., 1}_{n-2}, 2)$. Clearly, y belongs to the P-core of the game (N, V). But $\mathbf{y} \neq \mathbf{x}$, hence, y is not supported by an $\mathcal{S}$-equilibrium.

5 Strong and Coalition-proof Nash Equilibrium

In this section we formalize our assertion, stated in the Introduction, that the notion of $\mathcal{S}$-equilibrium is more demanding than Nash equilibrium: Not only single individuals are unable to better their situation by unilateral deviations, but also no group of players can correlate their actions in a way that is beneficial to them all. Indeed, we shall show in Theorem 3 that the set of $\mathcal{S}$-equilibria coincides with the set of strong Nash equilibria (Aumann (1959)) in a noncooperative game, G, which we shall define below. Theorems 1 and 3 imply that the set of strong Nash equilibrium of this game is nonempty (despite the fact that, as mentioned in the previous section, G is not a balanced game). Moreover, it turns out that the set of $\mathcal{S}$-equilibria of G coincides also with its set of coalition-proof Nash equilibria (Bernheim/ Peleg/ Whinston, 1987). This result is related to those obtained in Borm and Tijs (1991) and Borm and Peters (1991) who study the implementation of the core using strong and coalition proof Nash equilibrium. As is well-known, the set of strong Nash equilibria is, in general, a proper subset of the set of coalition-proof equilibria.

Let $\mathcal{S} = (N, \Omega, \phi, U)$ be a society with a finite number of individuals. Again, in view of the compactness of Ω and the continuity of each $u_i \in U$, we may assume, w.l.o.g., that $u_i(\omega) > 0$ for all $i \in N$ and all $\omega \in \Omega$. We associate with society $\mathcal{S}$ the following noncooperative game G which is closely related to the coalition formation game introduced by von Neumann and Morgenstern (1944). In this game each player chooses an alternative in Ω. If an alternative $\omega \in \Omega$ is feasible for the group C of all individuals who choose ω, then C forms and adopts ω. If ω is not feasible for C, i.e., $\omega \notin \phi(C)$, then each member of C receives a utility of 0.

Formally, the game G is defined as follows: The (finite) set of players is N and the strategy set of each $i \in N$ is Ω. Let $\mathrm{w} = (\omega_1, \omega_2, ..., \omega_n) \in \Omega^n$ be an n-tuple of strategies. Player i's payoff function, $h_i : \Omega^n \rightarrow R_+$, is given by:

$$h_i(\mathrm{w}) = \begin{cases} u_i(\omega_i) & if \quad \omega_i \in \phi(\{j \in N \mid \omega_j = \omega_i\}) \\ 0 & if \quad \omega_i \notin \phi(\{j \in N \mid \omega_j = \omega_i\}) \end{cases}$$

We recall

Definition 4: An n-tuple of strategies $\mathrm{w} = (\omega_1, \omega_2, ..., \omega_n)$ is a strong Nash equilibrium for the game G if there exist no coalition C and a vector of strategies $\mathrm{w}' \in \Re^C$ such that[7] $h_i(\mathrm{w}', \mathrm{w}^{N \setminus C}) > h_i(\mathrm{w})$ for all $i \in C$.

The n-tuple of strategies $\mathrm{w} = (\omega_1, \omega_2, ..., \omega_n)$ naturally generates the following partition of N:

$$P(\mathrm{w}) = \{C \subset N \mid \exists\, a \in \{\mathrm{w}\} \;\; s.t. \;\; C = \{i \in N \mid \omega_i = a\}\},$$

where $\{\mathrm{w}\}$ is the set of (distinct) coordinates of w. The relationship between a strong Nash equilibrium in the game G and an $\mathcal{S}$-equilibrium is determined by the following result.

Theorem 3: Let $\mathcal{S}$ be a society with a finite number of individuals. The n-tuple of strategies $\mathrm{w} = (\omega_1, ..., \omega_n)$ is a strong Nash equilibrium for the game G if only if (P, A) is an $\mathcal{S}$-equilibrium, where $P = P(\mathrm{w}) = \{C_1, C_2, ..., C_J\} \in \mathcal{P}$, $A = \{\mathrm{w}\} = \{a_1, a_2, ..., a_J\} \subset \Omega$, and $\omega_i = a_j$ for all $i \in C_j$, $j = 1, 2, ..., J$.

[7]For $T \subset N$, $\mathrm{y} \in \Re^T$, and $\mathrm{z} \in \Re^{N \setminus T}$, we denote by $\mathrm{x} = (\mathrm{y}, \mathrm{z}) \in \Re^N$ the vector whose coordinates satisfy $x^i = y^i$ for all $i \in T$ and $x^i = z^i$ for all $i \in N \setminus T$.

Proof of Theorem 3: The "if part". Let w be a strong Nash equilibrium and consider the pair $(P(\mathrm{w}), \{\mathrm{w}\})$. It is straightforward to verify that (D.1.1), (D.1.3)-(D.1.4) are satisfied. To check that the feasibility condition (D.1.2) also holds, note first that there exists a coalition $C_l \in P(\mathrm{w})$ such that $a_l \in \phi(C_l)$. Indeed, otherwise $h_i(\mathrm{w}) = 0$ for all $i \in N$. By (A.2), $\phi(N) \neq \emptyset$, thus there exists $a \in \phi(N)$. Consider the n-tuple of strategies w^a given by: $w_i^a \equiv a$ for all $i \in N$. Then, $h^i(\mathrm{w}^a) = u_i(a) > 0$ for all $i \in N$. Thus, $h^i(\mathrm{w}^a) > h^i(\mathrm{w})$ for all $i \in N$, contradicting our assumption that w is a strong Nash equilibrium. Now, if there were a coalition $C_k \in P(\mathrm{w})$ for which $a_k \notin \phi(C_k)$, then each member of C_k could increase his utility by adopting the strategy a_l. Hence, (D.1.2) also holds. Thus, $(P(\mathrm{w}), \{\mathrm{w}\})$ is an $\mathcal{S}$-equilibrium.

The "only if part". Let (P, A) be an $\mathcal{S}$-equilibrium, where $P = \{C_1, C_2, ..., C_J\}$, and $A = \{a_1, a_2, ..., a_J\}$. Define the n-tuple of strategies, $\mathrm{w} = (\omega_1, ..., \omega_n)$ by: $\omega_i = a_j$ for all $i \in C_j$, $j = 1, 2, ..., J$. Assume, in negation, that w is not a strong Nash equilibrium. Then, there exists a coalition C and vector of strategies $\mathrm{w}' = (\omega_i')_{i \in C} \in \Re^C$ such that $h_i(\mathrm{w}', \mathrm{w}^{N \setminus C}) > h_i(\mathrm{w})$ for all $i \in C$. Consider an alternative $\omega = \omega_k'$ for some $k \in C$. Let $T = \{i \in C \mid \mathrm{w}'_i = \omega\}$. By the definition of the game G, $h_i(\mathrm{w}', \mathrm{w}^{N \setminus C}) = u_i(\omega)$ for all $i \in T$. Then $u_i(\omega) > h_i(\mathrm{w})$ for all $i \in T$, yielding $\omega \in \phi(T)$, contradicting (D.1.4). Thus, w is, indeed, a strong Nash equilibrium.□

Theorems 1-3 and Example 1 imply that the set of strong Nash equilibria of the game G is, in general, a proper subset of the coalition structure core of the cooperative game studied in Section 4. This result differs from Kalai/ Postlewaite/ Roberts (1979) who determine the equivalence between the set of strong Nash equilibria and the coalition structure core, in their public goods model. This difference is due to the fact that, in contrast to us, they require strict monotonicity: Given a level of the public goods the utility of every individual $i \in C$ *strictly* increases when an individual outside of C joins this coalition. Within our framework this assumption is very restrictive and, therefore, we do not impose it.

We now turn to the notion of coalition-proof Nash equilibrium introduced by Bernheim/ Peleg/ Whinston (1987). In order to show that Theorem 3 remains valid also if we replace "strong" by "coalition-proof" Nash equilibrium, we use Greenberg's (1989 and 1991a) characterization of the latter notion. This characterization also points out the external stability of the set of strong Nash equilibrium in the coalition formation game G, (and hence, of the set of S-equilibria).

For a nonempty subset T of N and a vector $\mathrm{x} \in \Omega^n$ define the noncooperative game G_x^T as follows: The (finite) set of players is T and the strategy set of each $i \in N$ is Ω. The payoff of a player $i \in T$, in the game G_x^T, from the vector of strategies $\mathrm{w} \in \Re^T$ is defined to be $h^i(\mathrm{w}^T, \mathrm{x}^{N\setminus T})$. The set of strong Nash equilibrium of a game G is denoted $SNE(G)$.

Claim 3: If $\mathrm{x} \notin SNE(G)$ then there exist $C \subset N$ and $\mathrm{y}^C \in SNE(G_x^C)$ such that $h^i(\mathrm{y}^C, \mathrm{x}^{N\setminus C}) > h^i(\mathrm{x})$ for all $i \in C$.

Proof: By induction on the number of players, n. For $n = 1$ the claim is true since Ω is compact and h^i is continuous. Assume the validity of the claim for all games with less than n players, and consider the n-player game G. Let $\mathrm{x} \in \Omega^n \setminus SNE(G)$. If there exists $C \neq N$ such that $\mathrm{x}^C \notin SNE(G_x^C)$, then, by the induction hypothesis (on the game G_x^C which contains $|C| < n$ players) there exist $T \subset C$ and $\mathrm{y}^T \in SNE(G_x^T)$ with $h^i(\mathrm{y}^T, \mathrm{x}^{N\setminus T}) > h^i(\mathrm{x})$ for all $i \in T$, and we are done. It remains to consider the case where $\mathrm{x}^C \in SNE(G_x^C)$ for all $C \neq N$. Since $\mathrm{x} \notin SNE(G)$ and none of the proper subsets of N can improve upon x, it follows that the grand coalition can improve upon x. That is, there exists an n-tuple of strategies z such that $h^i(\mathrm{z}) > h^i(\mathrm{x})$ for all $i \in N$. Since Ω is compact and h^i is continuous, we can choose $\mathbf{z}$ to be Pareto optimal. Then $\mathbf{z} \in SNE(G)$. □

The above claim together with Greenberg (1989) and (1991a) immediately yield

Theorem 4: Let $\mathcal{S}$ be a society with a finite number of individuals. Then $\mathrm{w} = (\omega_1, ..., \omega_n)$ is a coalition-proof Nash equilibrium for the game G if only if (P, A) is an $\mathcal{S}$-equilibrium, where $P = P(\mathrm{w}) = \{C_1, C_2, ..., C_J\} \in \mathcal{P}$, $A = \{\mathrm{w}\} = \{a_1, a_2, ..., a_J\} \subset \Omega$, and $\omega_i = a_j$ for all $i \in C_j$, $j = 1, 2, ..., J$.

6 Extensions

The requirement in Theorem 1 that both N and Ω be finite sets cannot be dropped. Indeed, as the following example demonstrates, there exists a society $\mathcal{S}$ where both N and Ω contain an infinite number of elements which does not admit an $\mathcal{S}$-equilibrium. Moreover, this nonexistence remains true even when "partitions with an infinite number of subsets of N" are allowed to form.

Example 2: Let $N = \Omega = [0, 1]$. For each individual $i \in N$ the preference relation is represented by the distance from his peak $p^i = i$, i.e., for all $i \in N$ and all $\omega \in \Omega$

$$u_i(\omega) = -|\omega - i|.$$

The feasibility correspondence ϕ is given by[8]:

$$\phi(C) = \begin{cases} \Omega & \textit{if } C \textit{ contains a nonempty open interval} \\ \emptyset & \textit{otherwise} \end{cases}$$

[8]In atomless economies a coalition is defined to be a measurable (with respect to the Lebesgue measure on [0, 1]) subset of N. In contrast, in view of (A.2), here we must allow also for nonmeasurable sets which contain a (measurable) nonempty open interval.

Assume, in negation, that there exists $(P, A) \in E(S)$ (where P is not necessarily finite). Then, each coalition in P contains an open interval, for otherwise all its members get the utility level 0. (See the "if part" in the proof of Theorem 3.) Consider $C \in P$ and let $a(C) \in A$ be the alternative adopted by C in the equilibrium (P, A). Let $T = (\alpha, \beta)$, $0 \leq \alpha < \beta$ be such that $T \subset C$. Denote:

$$b_1 = \frac{2}{3}\alpha + \frac{1}{3}\beta, \;\; b_2 = \frac{1}{3}\alpha + \frac{2}{3}\beta, \;\; C_1 = (\alpha, b_1) \;\; and \;\; C_2 = (b_2, \beta).$$

Then, either $a(C) \leq \frac{\alpha+\beta}{2}$ or $a(C) \geq \frac{\alpha+\beta}{2}$, implying that

$$either \;\; u_i(b_1) > u_i(a(C)) \;\; \forall i \in C_1 \quad or \quad u_i(b_2) > u_i(a(C)) \;\; \forall i \in C_2.$$

Since for each $i \in C$, $u_i(a(C)) \geq u_i(a)$ for all $a \in A$, it follows that either C_1 or C_2 violate (D.1.4). Thus, S admits no equilibrium.

It is, however, possible to extend Theorem 1 if we impose the following "uniform continuity" assumption on the utility functions, restrict the feasibility correspondence, and somewhat relax the notion of equilibrium[9].

Assumption (A.5): The set of utility functions U is *equicontinuous*[10].

Definition 6: Let $\varepsilon > 0$ be given. A pair (P, A) is called an $\varepsilon - equilibrium$ if it satisfies (D.1.1) - (D.1.3) and, moreover,

(D.6.4) There exist no $C \subset N$ and $\omega \in \phi(C)$ such that $u_i(\omega) > u_i(a) + \varepsilon$ for all $i \in C$ and all $a \in A$.

That is, in an ε-equilibrium a new coalition will form only if each of its members is thereby made better off by at least ε units of (his) utils.

[9] We are thankful to one of the anonymous referees for pointing it out to us.

[10] The set of functions U is equicontinuous on the compact set Ω, if for any $\delta > 0$ there exists a $\sigma > 0$ such that $|u(a) - u(b)| < \delta$ for all $u \in U$ and all $a, b \in \Omega$ with $|a - b| < \sigma$.

Theorem 5: Let $\mathcal{S} = (N, \Omega, \phi, U)$ be a society that satisfies (A.1) - (A.3), (A.5), and where for all $C \subset N$, $\phi(C)$ is either Ω or else it is the empty set. Then, for every $\varepsilon > 0$, $\mathcal{S}$ admits an ε-equilibrium.[11]

Proof of Theorem 5: Let $\varepsilon > 0$ be given. Assumptions (A.1) and (A.5) yield the existence of a positive number $\sigma > 0$ such that $|u(a) - u(b)| < \varepsilon$ for all $u \in U$ and all $a, b \in \Omega$ with $|a - b| < \sigma$. (A.1) implies also that there exists a *finite* subset Q of Ω, such that for each $\omega \in \Omega$ there is $q \in Q$, satisfying $|\omega - q| < \sigma$, and, therefore, $|u_i(\omega) - u_i(q)| < \varepsilon$ for all for all $i \in N$. By Theorem 1, there exists a pair (P, A) which constitutes an equilibrium for the society $(\mathcal{S}/Q)$[12]. In order to show that (P, A) is an ε-equilibrium for $\mathcal{S}$ we only need to verify that (D.6.4) holds. Assume, in negation, that there exist a coalition C and $\bar{\omega} \in \phi(C)$, such that

$$u_i(\bar{\omega}) > u_i(a) + \varepsilon \ \ for \ \ all \ \ a \in A \ \ and \ \ all \ \ i \in C. \tag{1}$$

Since (P, A) is an equilibrium for $\mathcal{S}/Q$, it follows that $\bar{\omega} \notin Q$. However, the choice of Q guarantees that there exists $\bar{q} \in Q$ such that

$$|\ u_i(\bar{\omega}) - u_i(\bar{q})\ | < \varepsilon \ \ for \ \ all \ \ i \in N. \tag{2}$$

By (1) and (2), $\bar{q} \notin A$ and $C \subset S(\bar{q};\ A, \mathcal{S})$. Since $\bar{\omega} \in \phi(C)$, it follows that $\phi(C) \neq \emptyset$. Hence $\phi(C) = \Omega$ and, in particular, $\bar{q} \in \phi(C)$. By (A.2), $\bar{q} \in \phi(S(\bar{q}; A, \mathcal{S}))$, which, together with (2), contradict the fact that (P, A) is an equilibrium for $\mathcal{S}/Q$.$\square$

[11] Observe that Example 2 demonstrates that Theorem 5 is no longer true for $\varepsilon = 0$.
[12] The definition of the society $\mathcal{S}/Q$ is given after O.4 in Section 3.

7 References

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